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FLOW NEAR THE CRITICAL LINE IN THE GAS-DYNAMIC MODEL OF STELLAR WIND WITH THERMAL CONDUCTIVITY†

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A one-dimensional, time-independent, spherically symmetric model of stellar wind is studied within the framework of gas dynamics. It is assumed that the energy transfer in the external stellar atmosphere occurs by electron thermal conduction. The form of the critical line on which the transition from subsonic to supersonic flow occurs and the types of singular points and their connection with the directions of separatrices is studied in detail. The stability of the solutions with respect to perturbations in the neighbourhood of singular points is considered, using the method suggested in [1].

THE PROBLEM was first suggested in this formulation in [2] for two simplest cases: the adiabatic mode, that is essentially described by Parker's model [3], where, instead of the heat flux equation a politropic law was introduced, and the mode for which the total energy flow is equal to zero. For the latter one of the asymptotic forms at infinity was found. For the specified parameters the critical line was constructed and numerical calculations of the directions of separatrices were carried out [4].

1. The model considered is based on the following assumptions: the flow is one-dimensional, time-independent and spherically symmetric, the flow velocity has only a radial component, and all the parameters of the flow (the velocity u, pressure p, density ρ and temperature T) are the functions of only the space coordinate r, which is the distance from the centre of a star. The energy in the expanding stellar corona is transferred by electron thermal conduction. Viscosity effects and the influence of the magnetic field are not considered. The gas is assumed to be ideal with constant specific heats, and a ratio of the specific heats $\gamma = 5/3$.

With these assumptions the system of dynamic equations of the stellar atmosphere in dimensionless variables reduces to the following form

$$\frac{\psi}{2} - \lambda + \frac{5}{2}\tau + \frac{A}{2}\tau^{5/2}\frac{d\tau}{d\lambda} = \varepsilon_\infty$$

$$\frac{1}{2}\left(1 - \frac{\tau}{\psi}\right)\frac{d\psi}{d\lambda} = 1 - \frac{2\tau}{\lambda} - \frac{d\tau}{d\lambda} \tag{1.1}$$

$$\tau = \frac{T}{T_0}, \quad \psi = \frac{m_i u^2}{2kT_0}, \quad \lambda = \frac{GM_* m_i}{2kT_0 r}, \quad A = \frac{\kappa(T_0) GM_* m_i}{2ck^2 T_0}, \quad \varepsilon_\infty = \frac{E_\infty m_i}{2ckT_0}$$

Here $\kappa(T_0) = \kappa_0 T_0^{5/2}$, T_0 is the temperature at the base of the corona, G is the gravitational-constant, M_* is the stellar mass (assumed to be constant, as the contribution of the upper atmosphere to the stellar mass is unimportant), c is the mass flow rate constant, E_∞ is the constant value in the energy integral, k is

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Boltzmann's constant, m_i is the ion mass, and k is the thermal conductivity which depends on the temperature as [5]

$$\kappa = \kappa_0 T^{5/2} \approx 6\times 10^{-12} T^{5/2}\ \mathrm{J/(m\,s\,K)}$$

and serves as the electron thermal conductivity (the ion thermal conductivity, being of the order of m_e/m_i, where m_e is the electron mass, is negligibly small).

2. To study the singular points we will introduce a fictitious parameter λ of dimensionless time t, defined by the equation

$$dt/d\lambda = [\lambda A\tau^{5/2}(\psi-\tau)]^{-1} \tag{2.1}$$

As a result, we have the following system of three ordinary differential equations

$$\begin{aligned} d\tau/dt &= \lambda(\psi-\tau)(2\varepsilon_\infty - \psi + 2\lambda - 5\tau) = F_1(\tau,\psi,\lambda) \\ d\psi/dt &= 2\psi[A\tau^{5/2}(\lambda-2\tau) - \lambda(2\varepsilon_\infty - \psi + 2\lambda - 5\tau)] = F_2(\tau,\psi,\lambda) \\ d\lambda/dt &= \lambda A\tau^{5/2}(\psi-\tau) = F_3(\tau,\psi,\lambda) \end{aligned} \tag{2.2}$$

The substitution (2.1) eliminates singularities in system (1.1) in which $\lambda A\tau^{5/2}(\psi-\tau)=0$.

From the total set of singular points we select the singular points $(\tau_c,\ \psi_c,\ \lambda_c)$ that do not lie in the coordinate planes. These points are defined by the equations

$$\begin{aligned} &\psi_c = \tau_c,\quad \lambda_c(\tau_c,\varepsilon_\infty) = \zeta(\tau_c,\varepsilon_\infty) \pm [\zeta^2(\tau_c,\varepsilon_\infty) - A\tau_c^{7/2}]^{1/2} \\ &(\zeta(\tau_c,\varepsilon_\infty) = (6\tau_c + A\tau_c^{5/2} - 2\varepsilon_\infty)/4) \end{aligned} \tag{2.3}$$

and are traditionally called sonic though the equality $\psi=\tau$ means that $m_iu^2/(2kT)=1$ or the Mach number $M=\sqrt{(1/\gamma)}<1$.

The notation $\lambda_c=\lambda_c(\tau_c,\ \varepsilon_\infty)$ emphasizes the essential dependence of λ on ε_∞. For $\varepsilon_\infty=0$ the values of λ are defined on the half-line $\tau_c\in[0,\ +\infty)$, For $\varepsilon_\infty>0$ this is $\tau_c\in[\tau_c',\ +\infty)$, where τ_c is a unique solution of the equation $\zeta(\tau_c,\ \varepsilon_\infty)=\sqrt{(A)}\tau_c^{7/4}$ when $\tau_c\geqslant 0$. Higher ε_∞ values correspond to larger τ_c values. Moreover, for $\varepsilon_\infty=0$ the function λ_c can take any value from zero to infinity. For $\varepsilon_\infty>0$ a lower limit of λ_c' exists which corresponds to the greater value of ε_∞. Thus for $\varepsilon_\infty>0$ a transition through a singular point cannot occur too far from the centre of a star. In this case, the flow temperature must be sufficiently high.

Differentiation of the function $\lambda_c(\tau_c,\ \varepsilon_\infty)$ with respect to the parameter ε_∞ reveals that the curve corresponding to large ε_∞ is enclosed within the curve corresponding to lower ε_∞.The qualitative behaviour of the critical lines is shown in Fig. 1. Curves 1, 2 and 3 correspond to the functions $\lambda_c(\tau_c,\ 0)$, $\lambda_c(\tau_c,\ \varepsilon_\infty)$ $(\varepsilon_\infty>0)$ and $\lambda_c(\tau_c,\ \varepsilon_\infty')$ $(\varepsilon_\infty'>\varepsilon_\infty)$, respectively.

3. To find the nature of the singular points it is necessary to consider the characteristic matrix of system (2.2)

$$A_i^j = \partial F_i/\partial x_j|_{\tau_c,\psi_c,\lambda_c}$$

In this case, we assume that $(x_1,\ x_2,\ x_3)$ is the vector $(\tau,\ \psi,\ \lambda)$. We find the characteristic roots of this matrix $(z_1,\ z_2,\ z_3)$, that are the eigenvalues of system (2.2) and defined by the equation $\det\|A_i^j - z\delta_i^j\|=0$ or

$$\begin{aligned} &z(z^2 - G_2 z + G_1) = 0 \\ &G_2 = (2\tau_c + A\tau_c^{5/2})\lambda_c + 2A\tau_c^{7/2} \end{aligned} \tag{3.1}$$

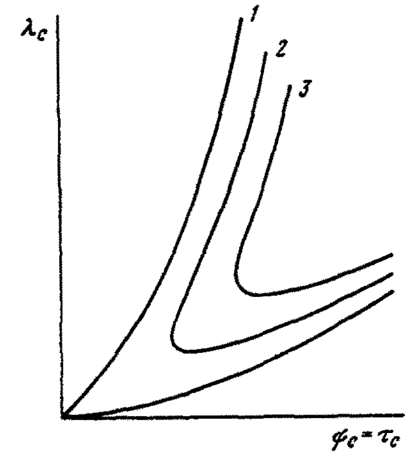

FIG. 1.

$$G_1 = -2A\tau_c^{7/2}[(4+5/2 A\tau_c^{3/2})\lambda_c^2 - 12(\tau_c + A\tau_c^{5/2})\lambda_c + 16A\tau_c^{7/2}]$$

The singular points (2.3) fill in the continuous set which is a manifold of unit dimensions. Thus [6], at such singular points system (2.2) has one zero eigenvalue. In this case, the singular points (2.3) are characterized as two-dimensional singular points if the set of singular points (2.3) is not degenerate, i.e. when almost at any singular point of this set there are two eigenvalues of system (2.2) with non-zero real parts. Hence it is necessary to study the equation corresponding to the case when the expression in parentheses in Eq. (3.1) is equal to zero.

If the eigenvalues z_1 and z_2 (the roots of this equation) are complex conjugate, i.e. the discriminant D of the equation is negative, then for such parameters the singular point is a focus. If the eigenvalues are real, different and have like signs (for $D > 0$, $G_1 > 0$), the singular points are nodes. If the eigenvalues are real but have unlike signs (for $D > 0$, $G_1 > 0$), the singular points are saddles. In the case of $D = 0$ the singular points are degenerate nodes. When $G_1 = 0$ they are degenerate saddles.

The relative position of the curves $D = 0$, $G_1 = 0$ and the critical lines (the dashed curve) for $A = 400$, that corresponds to the parameters of the solar wind, is shown in Fig. 2.

If the singular point situated on the line $\lambda_c(\tau_c, \varepsilon_-)$ falls into the domain 1, this point is a focus. The integral curves in the neighbourhood of such a singular point have no physical meaning. If the singular point falls into the domain 2, the node corresponds to it. At the intersection of domains 1 and 2 the singular points will be degenerate nodes. In domain 3 the singular points are nodes. At the intersection of domains 2 and 3 these points will be degenerate saddles.

4. To define the separatrices at the singular points $(\tau_c, \psi_c, \lambda_c)$ it is necessary to find the coefficients α and β of the decomposition

$$\tau - \tau_c = \alpha(\lambda - \lambda_c), \qquad \psi - \tau = \beta(\lambda - \lambda_c) \tag{4.1}$$

When $\beta > 0$ the transition from supersonic to subsonic flow occurs when the radial distance increases, and $\beta > 0$ corresponds to the transition from subsonic to supersonic flow.

Substituting expressions (4.1) into system (1.1) and considering the infinitesimal terms of the zeroth order with respect to $\lambda - \lambda_c$ we obtain two expressions for α that are identical in accordance with (2.3)

$$A\tau_c^{5/2}\alpha = 2\lambda_c - 6\tau_c + 2\varepsilon_\infty, \qquad \alpha = 1 - 2\tau_c / \lambda_c \tag{4.2}$$

Therefore, in order to obtain expressions for β it is necessary to consider a higher-order term of the decomposition of $\tau - \tau_c$ up to and including the quadratic term.

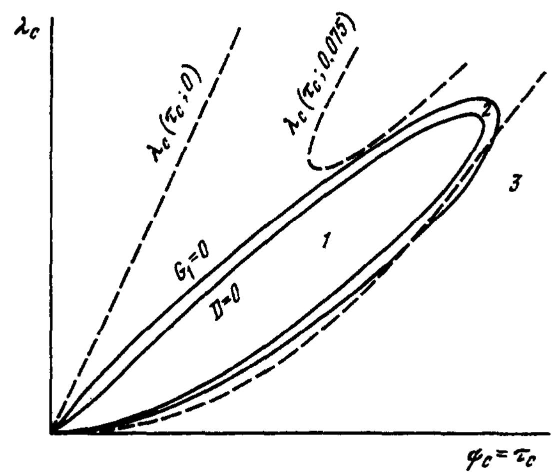

FIG. 2.

This means that the integral curves in the neighbourhood of the singular point do not lie in a plane

$$\tau-\tau_c=\alpha(\lambda-\lambda_c)+\delta(\lambda-\lambda_c)^2 \tag{4.3}$$

Substituting expressions (4.3) and the second expression of (4.1) and then considering the infinitesimal terms of the first order with respect to $\lambda-\lambda_c$ we obtain two algebraic equations for β and δ. Eliminating δ from this system we obtain a quadratic equation for β

$$A\tau_c^{3/2}\lambda_c^2\beta^2-\lambda_c\tau_c^{-1}G_2\beta+G_1/(A\tau_c^{7/2})=0 \tag{4.4}$$

Since the discrimination of Eq. (4.4) is equal to $\lambda_c^2\tau_c^{-2}D$ the singularities such as a focus correspond to the complex conjugate roots of this equation. When the roots of Eq. (4.4) are real and have unlike signs the singularities are saddle points. In this case, one of the separatrices passes from the subsonic to the supersonic domain, and another passes from the supersonic to the subsonic domain.

If the roots are real and have like signs, the singular points are nodes. Since the sum of the roots is equal to $G_2/(A\tau_c^{5/2}\lambda_c)$ and at $D>0$ and $G_1>0$ the function G_2 is positive, both of the separatrices passing through the node singular point correspond to the transition from supersonic to subsonic flow.

In the case of a degenerate node and a degenerate saddle, the only separatrix passing through any such a singular point corresponds to the transition from supersonic to subsonic flow when the radial distance is increased.

Hence, the transition from subsonic to supersonic flow occurs only at the saddle point along one of the separatrices.

5. Among the whole set of the integral curves in the neighbourhood of the $\psi=\tau$ plane the transition from subsonic to supersonic flow is only possible at saddle singular points with one separatrix at every such point. It is precisely this case that occurs in stellar and solar wind.

In the neighbourhood of singular points such as a focus the solutions are not realizable physically since they are not continuous with respect to λ

Since the node singular points have separatrices passing from the supersonic to the subsonic domain, all the integral curves in the neighbourhood of such a point are unstable to perturbations in the neighbourhood of the singular point [1]. For the saddle singular points only one integral curve is stable to perturbations in the neighbourhood of the singular point, namely, the curve which is a separatrix for the transition for subsonic to supersonic flow. All the remaining integral curves are absolutely unstable.

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